Chi-Squared Analysis of the Uncertainty of Falling Chain Experiments

By David Hu

Author Bio

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Abstract

The falling chain problem is a classic physics problem that studies the dynamics of a chain falling in a gravitational field. There are different ways to investigate the problem. One of them is a U-Chain system in which a chain is attached to a rigid support by its two ends. As one of the ends is released, the chain begins to fall. Various models have been suggested to explain the acceleration of the falling end in a U-Chain system, but few of these analyses have determined a best-fit model and fewer still have tested their models. In this project, a one-parameter chi-squared analysis is used to test a model that describes the relationship between the falling time of one end of U-chains and their lengths. The falling of five chains is recorded and used in the experiment. A best-fit model was obtained, but was not a good fit to the data, despite only a small difference between the best-fit and model-predicted parameter values. The lack of a good fit is attributed to underestimated measurement uncertainties due to the measurement method. The fact that a good fit was not obtained despite only a small difference between best-fit and model-predicted parameter values is evidence of the sensitivity of chi-squared model testing.

Keywords: Falling Chain, Chi-Squared Analysis, Uncertainty, Best-Fit Model, Kinetic Energy, Potential Energy, Goodness of Fit, Standard Error

Introduction

The falling chain problem was studied as early as the 19th century (Cayley, 1857). There are two well-known falling chain problems, U-chain and pile-chain (de Sousa et al., 2012). In the U-chain system, a flexible heavy chain is attached to a rigid support by its two ends. As one of the ends is released, the chain begins to fall. It is found that if the distance between the two ends is small compared with the length of the chain, the falling end will accelerate faster than gravitational acceleration g. In the pile-chain system, a chain is compacted together in a heap and one end is brought over the edge of a table and starts falling with an acceleration of g/3 (de Sousa et al., 2012; Tomaszewski et al., 2006).

The falling chain problem can be studied using chains with different lengths, different weights, or both. Wong and Yasui (2006) indicated that energy conservation arises because the links are transferred by elastic collisions as opposed to inelastic collisions. They found that the end of a falling chain accelerates faster than g due to the part of the chain right under the falling end which pulls the falling end down further. Géminard and Vanel (2008) analyzed the tip of a freely falling chain in a gravitational field where one end is suspended and concluded that the time-resolved measurements of horizontal and vertical components complement previous laboratory and simulations of free-end dynamics. Clouser and Oberla's experiment (2008) showed that energy is not conserved due to inelastic collisions. De Sousa et al. (2012) analyzed the theoretical and experimental data of a falling U-chain and a pile-chain. They found that a two sub-system model based on energy conservation provided a good fit to the data. Pantaleone (2019) developed a simple model to measure the size of the tension that pulls the chain downwards based on the length and width of a link, the maximum beading angle at a junction, and the inclination angle of the surface and the coefficients of friction and restitution between the chain and the surface.

While most of the previous studies have centered on the explanation of the acceleration of different falling chains using different theories in physics, few studies have considered if the model represents the data well and if the data collected is good enough to represent the phenomenon that we are studying. These questions are about the randomness and uncertainty in the process of data collecting and modeling in physics experiments. Different methods have been used to evaluate modeling results. Chi-squared (X2) analysis is one of them.

Chi-squared analysis is a statistical analysis used to compare actual results with an expected hypothesis. It combines both curve-fitting and model-testing by considering measurement uncertainties. Hence, chi-squared analysis can estimate model parameter values and test a model's goodness-of-fit. The two most used versions of this analysis method are the chi-square goodness of fit test and the chi-square test of independence. Witkov and Zengel (2019) used chi-squared analysis to test a one-parameter linear model (y=Ax) that estimated the ratio of the maximum force when a chain is falling onto a scale to the weight of the chain. In their study, y is the maximum force. x is the weight of the chain. A is 3. That is, in a theoretical condition, when a chain falls on a scale, the maximum force is 3 times the weight of the whole chain. One-parameter chi-squared analysis was applied on the data collected by the authors. A best fit model of y=2.96x was obtained, very close to the theoretical model. The result of the chi-squared analysis also indicated that the model was a good fit because it fell within two standard errors of the best fit parameter. The case study proved that chi-squared analysis is a very useful tool to evaluate models and uncertainties in data collection. Currently, no study has used chi-squared analysis to test models regarding the fall time in falling chain problems.

Therefore, in this project, the chi-square goodness of fit analysis was applied to evaluate a model of a U-chain system. The objectives of this study are to obtain a best-fit parameter value (i.e., slope) for a theoretical model and to examine if the best-fit is a good fit to the collected data.

Research Methodology

A Theoretical Model of U-Chain Falling System

In this project, a U-chain system is used. That is, a chain is attached to a rigid support by its two ends. As one of the ends is released, the chain begins to fall (Figure 1). According to C. Witkov (personal communication, July 24, 2021), a theoretical model has been derived that represents the relationship between the length of the chain (L) and the time the falling end reaches the lowest point (T). This theoretical model is based on the potential and kinetic energy of both sides of the chain. The main steps of the model derivation are as below:

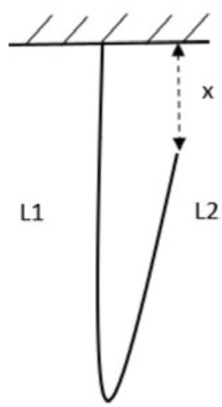

Figure 1. Schematic of a falling chain

L_1: *the length of the left portion of the chain*
L_2: *the length of the right portion of the chain*
x: *the distance between the rigid support and the falling end*

Step 1: When both ends are attached to the rigid support, the left and right portion have the same length.

$$L_1 = L_2 = \frac{L}{2} \tag{1}$$

Step 2: When the right end of the chain starts falling, the length of the left side will be increasing, and the right side will be decreasing.

$$L_1 = \frac{L}{2} + \frac{x}{2} = \frac{L+x}{2} \tag{2}$$

$$L_2 = \frac{L}{2} - \frac{x}{2} = \frac{L-x}{2} \tag{3}$$

Step 3: When the right end is falling, the potential energy of both sides will also be changing.

$$U_1 =- m_1 g h_1 \tag{4}$$

$$U_2 =- m_2 g h_2 \tag{5}$$

where, U_1 and U_2 are the potential energy of the left and right portion of the chain respectively, m_1 and m_2 are the mass of the left and right portion of the chain respectively, g is the gravity of the Earth, and h_1 and h_2 are the center of the mass of the left and right respectively. m_1, m_2, h_1,and h_2 can be decided using the following equations:

$$m_1 = \mu L_1 = \frac{\mu(L+x)}{2} \tag{6}$$

$$m_2 = \mu L_2 = \frac{\mu(L-x)}{2} \tag{7}$$

where, μ is the linear density.

$$h_1 = \frac{1}{2}L_1 = \frac{\left(\frac{1}{2}\right)(L+x)}{2} \tag{8}$$

$$h_2 = x + \frac{1}{2}L_2 = x + \frac{\left(\frac{1}{2}\right)(L-x)}{2} \tag{9}$$

Then,

$$U_1 =- \left(\frac{\mu g}{8}\right)\left(L^2 + 2Lx + x^2\right) \tag{10}$$

$$U_2 =- \left(\frac{\mu g}{8}\right)(2Lx + L^2 - 3x^2) \tag{11}$$

So, the total potential energy of both sides, Utotal, is as below:

$$U_{total} = U_1 + U_2 =- \left(\frac{\mu g}{4}\right)(L^2 + 2Lx - x^2) \tag{12}$$

Step 4: As the right side of the chain is moving, there is also the kinetic energy on the right side calculated as below:

$$K_2 = \frac{1}{2} m_2 v^2 = \frac{\mu(L-x)}{4} v^2 \tag{13}$$

where K2 is the kinetic energy of the right portion of the chain, and v is the velocity of the falling end of the chain, and $v = \frac{dx}{dt}$

Step 5: Based on the law of energy conservation, the initial energy (Ei) equals the final energy (E_f). And the total energy (E) at any given time is the sum of kinetic energy (K) and potential energy (U). Therefore,

$$K_i + U_i = K_f + U_f \tag{14}$$

where, K_i and K_f are the initial and final kinetic energy respectively, and U_i and U_f are the initial and final potential energy respectively.

As the initial kinetic energy is 0 when x=0, the above equation can be written in the following format:

$$0 + \left(-\frac{\mu g}{4}\right)\left(L^2\right) = \left(\frac{\mu}{4}\right)(L - x)v^2 + \left(-\frac{\mu g}{4}\right)\left(L^2 + 2Lx - x^2\right) \tag{15}$$

Then,

$$(L - x)v^2 = 2gLx - gx^2 \tag{16}$$

$$v = \frac{dx}{dt} = \sqrt{gL}\sqrt{\frac{2\left(\frac{x}{L}\right)-\left(\frac{x}{L}\right)^2}{1-\frac{x}{L}}} \tag{17}$$

$$\frac{dx}{\sqrt{\frac{2\left(\frac{x}{L}\right)-\left(\frac{x}{L}\right)^2}{1-\frac{x}{L}}}} = \sqrt{gL}\, dt \tag{18}$$

Step 6: Derive the theoretical model by integrating both sides of the above equation.

$$\int_0^1 \sqrt{\frac{1-\frac{x}{L}}{\left[2\left(\frac{x}{L}\right)- \left(\frac{x}{L}\right)^2\right]}}\, d\left(\frac{x}{L}\right) = \int_0^{t_F} \sqrt{\frac{g}{L}}\, dt \tag{19}$$

where, t_F is the final time. Then,

$$1.19814 = \sqrt{\frac{g}{L}}\, t_F \tag{20}$$

Since g=9.8 m/s2 , the theoretical model is:

$$t_F = 0.38\sqrt{L} \tag{21}$$

Videos of falling chain experiments

Five videos of a falling chain process were recorded. Each video used a different chain length. The original video is 400 frames per second (fps). To be able to measure the time of the falling chains, the videos were converted to a slow motion of 30 fps.

Timing strategy

Time was measured between the start and end of the falling process using the stopwatch function on a commercially available cell phone. Five repeated and independent measurements were performed for each of the five falling chain videos. Since the frame rate of the video is not real-time, actual time was calculated using Equation 22.

$$Actual\ time = \left(\frac{30fps}{400fps}\right) video\ time\ from\ stopwatch \tag{22}$$

Apply the above data in chi-squared goodness of fit analysis

Chi-squared goodness of fit analysis is based on a Gaussian distribution of uncertainties around each data point and the central limit theorem (Witkov & Zengel, 2019). The equation of chi-squared analysis is defined as

$$X^2 = \frac{\Sigma\left(y_i - y_{model_i}\right)^2}{\sigma_i^2} \tag{23}$$

where, σ_i is the standard error of the mean in the measurement, representing the uncertainty of the ith measurement. depends on the number of measurements (N) and the root-mean-square deviation (RMSD) of the data.

$$\sigma_i = \frac{RMSD}{\sqrt{N}} \tag{24}$$

In the case of a one-parameter linear model, the X^2 is defined as

$$X^2 = \frac{\sum_i \left(y_i - Ax_i\right)^2}{\sigma_i^2} \tag{25}$$

The best fit value of the slope A is the one that minimizes X^2. But, the best fit model is not necessarily a good fit to the data. Chi-squared analysis assumes that uncertainties on x are negligible and only the uncertainties on y are evaluated. Considering the uncertainty in measurements, a model is a good fit to the data if most of the measurements are within one of the line of best fit. The minimum value of X^2 (X^2_{min}) can then be determined based on one σ from the best-fit model. If

$$y_i = y_{model_i} \pm 1\sigma_i = Ax_i \pm 1\sigma_i \quad (26)$$

Then,

$$X^2 = \frac{\sum_i^N (y_i - Ax_i)^2}{\sigma_i^2} = \frac{\sum_i^N (Ax_i \pm 1\sigma_i - Ax_i)^2}{\sigma_i^2} = \sum_i^N 1 = N \quad (27)$$

According to Equation 27, if a model is a good fit to the data, the X^2_{min} should be around the order of N.

$$X^2_{min} \approx N \text{ for a good fit} \quad (28)$$

Therefore, any value that is within N±2N is a good value of X^2_{min}.

Chi-squared analysis also provides an uncertainty estimate for the best fit parameter. A model will be rejected if the parameter A in the theoretical model is more than two standard deviations (Aσ) of the Gaussian distribution of the best fit parameter value (A_{best}). That is, the uncertainty of Abest is represented as A_{best}±2Aσ. This uncertainty can be found by examining the interception between $X^2_{min}+4$ and a $X^2(A)$ parabola.

To test the model and data collected in this project, the chains' lengths and their falling times from the above steps were brought into the script of one-parameter chi-squared analysis developed by Witkov and Zengel (2017).

Results

The five chains' lengths, the falling times from the stopwatch, and the actual times each chain took to fall to the bottom are shown in Table 1.

	Chain 1		**Chain 2**		**Chain 3**		**Chain 4**		**Chain 5**	
Length	0.781m		1.076m		1.314m		1.596m		1.841m	
Time (s)	**Video Time**	**Actual Time**	**Video Time**	**Actual Time**	**Video Time**	**Actual Time**	**Video Time**	**Actual Time**	**Video Time**	**Actual Time**
Time 1	4.35	0.326	4.66	0.350	5.46	0.410	6.16	0.462	7.05	0.529
Time 2	4.39	0.329	4.71	0.353	5.53	0.415	6.08	0.456	6.87	0.515
Time 3	4.34	0.326	4.75	0.356	5.57	0.418	6.10	0.458	7.01	0.526
Time 4	4.22	0.317	4.74	0.356	5.60	0.420	6.15	0.461	6.86	0.515
Time 5	4.40	0.330	4.77	0.358	5.49	0.412	6.04	0.453	6.98	0.524

The chains' lengths are filled in the script as the x-variable arrays (independent variable) and the five measurements of each chain's falling time are used as the y-variable arrays (dependent variable) (Table 2).

Table 2 Filling variable arrays in the script

Variable arrays	Values
Independent variable (x)	[0.781, 1.076, 1.314, 1,596, 1.841]
Dependent variable 1 (y)	[0.326, 0.329, 0.326, 0.317, 0.330]
Dependent variable 2 (y)	[0.350, 0.353, 0.356, 0.356, 0.358]
Dependent variable 3 (y)	[0.410, 0.415, 0.418, 0.420, 0.412]
Dependent variable 4 (y)	[0.462, 0.456, 0.458, 0.461, 0.453]
Dependent variable 5 (y)	[0.529, 0.515, 0.526, 0.515, 0.524]

The results from chi-squared analysis (Figure 2) indicate that the best fit parameter A_{best} is 0.36 and the minimum value of X^2 is 329.59.

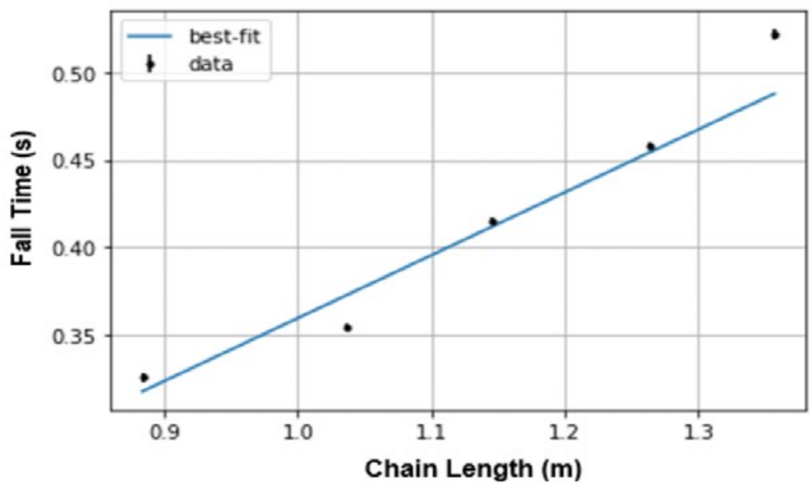

Figure 2
Falling chain data plot

Running the script of chi-squared analysis also created a X2 plot (Figure 3):

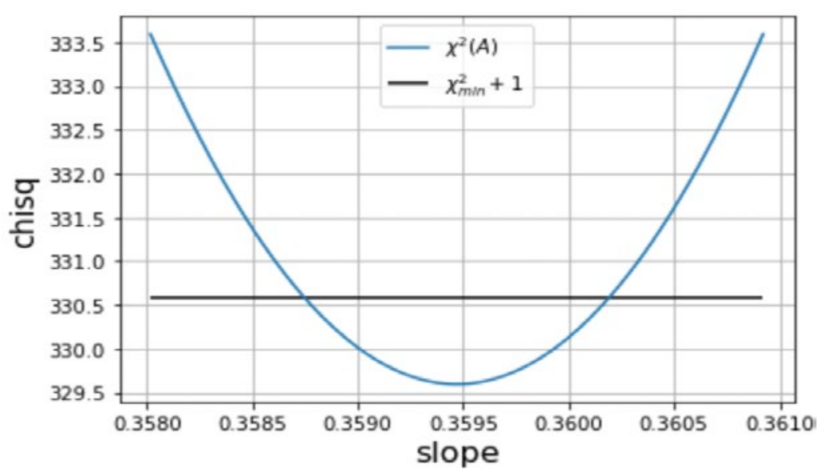

Figure 3
Falling chain X^2 plot

Discussion and Conclusion

This study represents the first research applying chi-squared analysis to the classic problem of falling U-chain. Running the chi-squared analysis script gives the best-fit parameter A_{best}=0.36, indicating that the model y=0.36x is a best-fit to the data. However, the X^2_{min} is 329.59, far beyond the required range of a good fit, which is between 1.84 and 8.16 $(N \pm \sqrt{2N})$. This means that this model is not a good fit to the data collected.

A major reason of the high value of X^2_{min} is that measurement uncertainties are underestimated in this experiment due to the measurement method such as human recording of a stopwatch and measuring time from videos instead of actual chain falling. From the definition of chi-squared analysis, underestimation of uncertainties leads to overestimation of chi-squared values (including the minimum value of chi-squared).

As a result of chi-squared analysis including measurement uncertainties in its definition, one of the attractive features of chi-squared analysis is that failure to meet its good-fit criteria reveals information not only about the model, but also about the experiment used to test the model.

The close numerical match between the best-fit parameter value (A_{best}=0.36) and the theoretical parameter value (A=0.38) despite failing the goodness-of-fit test demonstrates the sensitivity of chi-squared model testing to both the model and correctly estimating measurement uncertainties.

Based on the inspection of the interception between $X^2_{min}+4$ and the $X^2(A)$ parabola in Figure 3, the uncertainty of A_{best}, represented as $A_{best} \pm 2\sigma_A$, is determined to be between 0.3580 and 0.3609, which does not contain the parameter A of the theoretical model, 0.38. This further confirms that the model is rejected.

This project demonstrates that chi-squared analysis is a very useful tool for model testing and uncertainty measurement. In this work, chi-squared analysis is applied to the falling chain problem. The results indicate that the model is a best-fit, but not a good fit to the data, and that high uncertainties exist in the measurement of A_{best}. To improve the model fit in a future study, the measurement method may be modified, such as testing the falling time directly from a falling chain instead of using videos, using high-resolution videos, using some auto-tracking programs to measure falling time, and increasing the number of chains used in the experiment. Furthermore, air resistance may be considered in the derivation of the theoretical model.

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