

The Kissing Number Problem in Three Dimensions: Geometric and Linear Programming Methods

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ABSTRACT

The kissing number problem asks for the largest number of congruent, nonoverlapping spheres that can touch a fixed sphere of the same radius. This literature review studies the three-dimensional case and asks how geometric and optimization-based methods establish the equality $k(3) = 12$. The analysis first converts a kissing configuration into a spherical code with minimum angular separation 60° . It then distinguishes the lower-bound construction supplied by the face-centered cubic lattice from the substantially harder upper-bound problem. The geometric approach is summarized through spherical caps, contact graphs, and the Schütte-van der Waerden exclusion of a thirteen-point code. The linear-programming approach is developed from Schoenberg positivity, the Delsarte bound, and Musin's degree-nine auxiliary polynomial together with its required local spherical analysis. The comparison shows that geometric proofs expose the low-dimensional structure of extremal configurations, whereas linear and semidefinite programming provide a systematic framework that extends more effectively across dimensions. The principal finding is that neither packing density nor an explicit twelve-sphere construction alone proves optimality; the decisive step is an upper-bound that combines global algebraic positivity with local geometric constraints.

Keywords: *kissing number, sphere packing, spherical code, discrete geometry, face-centered cubic lattice, Gram matrix, linear programming, Legendre polynomials, Delsarte method, semidefinite programming*

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THE KISSING NUMBER PROBLEM IN THREE DIMENSIONS: GEOMETRIC AND LINEAR-PROGRAMMING METHODS

The kissing number $k(n)$ is the maximum number of congruent, nonoverlapping n -dimensional spheres that can simultaneously touch a fixed sphere of the same radius in R^n . In three dimensions, twelve spheres can easily be placed around a central sphere, for instance, with centers corresponding to vertices of an icosahedron as a configuration, but the visible gaps between them make it far from obvious that a thirteenth sphere is impossible. This distinction between constructing twelve spheres and proving that no larger configuration exists is the central logical difficulty of the problem.

The three-dimensional question is traditionally associated with the Newton-Gregory dispute of 1694. Newton asserted that twelve was maximal, whereas Gregory believed that thirteen might fit. The first generally accepted proof that $k(3) = 12$ was given by Schütte and van der Waerden in 1953, followed by a concise proof sketch by Leech in 1956 (Leech, 1956, pp. 22-23; Schütte & van der Waerden, 1953, pp. 325-334). Musin later supplied a different proof that combines linear-programming ideas with elementary spherical geometry (Musin, 2006, pp. 375-384).

The problem also belongs to the broader theory of spherical codes. In information theory, points in high-dimensional Euclidean space represent possible transmitted signals, and minimum distance controls resistance to noise (Shannon, 1948, pp. 379-423). Kissing numbers therefore measure a local packing constraint that is mathematically related to error-correcting codes, lattice theory, harmonic analysis, and optimization.

This paper asks: How do geometric and linear-programming methods prove and generalize the result $k(3) = 12$? The method is a comparative mathematical review. The paper first derives the spherical-code formulation, then constructs twelve contacts using the face-centered cubic lattice, summarizes the geometric upper-bound strategy, develops the Delsarte and Musin inequalities, and compares the logical strength and generalizability of the methods.

DEFINITIONS AND PRELIMINARIES

Let the central sphere and all surrounding spheres have radius 1. If the central sphere is centered at the origin and a surrounding sphere has center c_i , tangency to the central sphere gives

$$\|c_i\| = 2.$$

Define $x_i = c_i/2$. Then $x_i \in S^{n-1}$, where
 $S^{n-1} = \{x \in R^n : \|x\| = 1\}.$

Two surrounding unit spheres do not overlap precisely when $\|c_i - c_j\| \geq 2$. After scaling, this becomes $\|x_i - x_j\| \geq 1$. Since $\|x_i\| = \|x_j\| = 1$,

$$\|x_i - x_j\|^2 = \|x_i\|^2 + \|x_j\|^2 - 2\langle x_i, x_j \rangle = 2 - 2\langle x_i, x_j \rangle.$$

Therefore,

$$\|x_i - x_j\| \geq 1 \Leftrightarrow \langle x_i, x_j \rangle \leq \frac{1}{2}.$$

Writing ϕ_{ij} for the angular distance between x_i and x_j , the identity $\langle x_i, x_j \rangle = \cos \phi_{ij}$ yields

$$\phi_{ij} \geq \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

A spherical code is a finite set $X \subset S^{n-1}$ whose distinct points satisfy a prescribed minimum angular separation. If $A(n, \theta)$ denotes the largest possible size of a code in S^{n-1} with minimum angle at least θ , then the scaling argument proves the exact equivalence

$$k(n) = A\left(n, \frac{\pi}{3}\right).$$

This formulation separates the problem into a lower bound and an upper bound. An explicit code of size N proves $k(n) \geq N$. To prove equality, one must also show that every admissible code has at most N points.

Table 1
Exact Kissing Numbers in the Dimensions Currently Solved

Dimension n	Exact value $k(n)$	Principal structure or method
1	2	Two endpoints
2	6	Regular hexagon
3	12	Geometric proof; extended linear programming
4	24	The 24-cell; extended linear programming
8	240	Minimal vectors of the E_8 lattice
24	196,560	Minimal vectors of the Leech lattice

Note. Created by author, 2026, from Musin (2006, 2008) and Odlyzko and Sloane (1979).

HISTORICAL DEVELOPMENT

A simple area estimate helps explain why Gregory's conjecture appeared plausible. Around each point of a spherical code with minimum angle 60° , place a spherical cap of angular radius 30° . These caps have

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disjoint interiors. A cap of angular radius α on the unit sphere has area $2\pi(1 - \cos\alpha)$, so the area argument gives

$$N 2\pi\left(1 - \cos\frac{\pi}{6}\right) \leq 4\pi.$$

Consequently,

$$N \leq \frac{2}{1-\sqrt{3}/2} \approx 14.93,$$

and hence $N \leq 14$. This estimate is rigorous but too weak to distinguish twelve from thirteen. Its failure illustrates a recurring theme: total area does not encode how caps interact combinatorially on the sphere.

Schütte and van der Waerden resolved the problem by analyzing the geometry of a hypothetical thirteen-point spherical code. Their proof reduces the possible configurations through spherical trigonometry and shows that the required angular separations cannot be maintained (Schütte & van der Waerden, 1953, pp. 325-334). Leech's later note compressed the argument but did not change its fundamentally geometric character (Leech, 1956, pp. 22-23). Musin's proof replaced much of the case structure with an auxiliary polynomial while retaining a local geometric argument for exceptional nearly antipodal pairs (Musin, 2006, pp. 375-384).

GEOMETRIC AND LATTICE METHODS

The FCC Construction and the Lower Bound

The face-centered cubic lattice can be represented as

$$D_3 = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1 + x_2 + x_3 \equiv 0 \pmod{2}\}.$$

One basis is

$$b_1 = (1, 1, 0), \quad b_2 = (1, 0, 1), \quad b_3 = (0, 1, 1).$$

Let B be the matrix whose columns are these basis vectors. Then

$$B = (1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1), \quad G = B^T B = (2 \ 1 \ 1 \ 1 \ 2 \ 1 \ 1 \ 1 \ 2).$$

The fundamental volume is

$$|\det B| = \sqrt{\det G} = 2.$$

The shortest nonzero lattice vectors have squared norm 2. They are the twelve vectors obtained by permuting the coordinates of $(\pm 1, \pm 1, 0)$. If spheres are centered at lattice points, the largest nonoverlapping radius is therefore $r = 1/\sqrt{2}$. Each sphere touches exactly the twelve spheres whose centers differ by a shortest vector. Thus the FCC lattice proves $k(3) \geq 12$.

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The packing density is

$$\Delta_{FCC} = \frac{\frac{4}{3}\pi(1/\sqrt{2})^3}{2} = \frac{\pi}{3\sqrt{2}} = \frac{\pi}{\sqrt{18}}.$$

Hales proved that no packing of congruent spheres in R^3 has density greater than this value, and the later Flyspeck project formally verified the proof (Hales, 2005, pp. 1065-1185; Hales et al., 2017). However, the Kepler conjecture and the kissing number problem optimize different quantities. Packing density is a global average, whereas the kissing number is a local maximum around one sphere. Therefore, the optimal density of FCC does not imply the upper bound $k(3) \leq 12$. The FCC calculation supplies only the construction side of the kissing-number proof.

Geometric Upper-Bound Strategy

For an admissible code $X \subset S^2$, form a contact graph whose vertices are the points of X and whose edges join pairs at angular distance exactly 60° . The graph is drawn by shorter great-circle arcs. Distinct edges cannot cross in their interiors, because a crossing would force one pair of endpoints to be separated by less than 60° . The contact graph is therefore planar.

Planarity introduces Euler's formula

$$V - E + F = 2,$$

and, when all faces have at least three edges, the usual inequality $E \leq 3V - 6$. Also, for a spherical triangle with side lengths a, b, c and angle C opposite c ,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The spherical cosine law thus converts angular-separation conditions into restrictions on face angles and diagonals. In a thirteen-point code, one studies locally rigid contact graphs and shows that every surviving combinatorial pattern violates at least one metric inequality.

The strength of this approach is its transparency in low dimension. It identifies why area estimates fail, shows how local contacts constrain global organization, and reveals the role of spherical polygons. Its weakness is generalizability. As dimension increases, planar contact graphs disappear, the classification of local configurations becomes more difficult, and spherical trigonometric case analysis grows rapidly.

LINEAR-PROGRAMMING METHODS

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Schoenberg Positivity

Let $X = \{x_1, \dots, x_N\} \subset S^2$. The Legendre polynomials satisfy

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{1}{2}(3t^2 - 1),$$

and, more generally, its explicit formula is as follows

$$P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k.$$

Schoenberg's theorem implies that for every $k \geq 0$,

$$\sum_{i=1}^N \sum_{j=1}^N P_k(\langle x_i, x_j \rangle) \geq 0$$

(Schoenberg, 1942, pp. 96-108). This positivity is the algebraic foundation of the Delsarte method developed for spherical codes by Delsarte et al. (1977, pp. 363-388).

The Classical Delsarte Bound

Suppose

$$f(t) = \sum_{k=0}^d f_k P_k(t)$$

satisfies $f_0 > 0$, $f_k \geq 0$ for $k \geq 1$, and $f(t) \leq 0$ for $-1 \leq t \leq s$. For a code whose distinct inner products are at most s , define

$$S_f(X) = \sum_{i=1}^N \sum_{j=1}^N f(\langle x_i, x_j \rangle).$$

By Schoenberg positivity,

$$S_f(X) = \sum_{k=0}^d f_k \sum_{i,j} P_k(\langle x_i, x_j \rangle) \geq f_0 N^2.$$

On the other hand, the sign condition makes every off-diagonal term nonpositive, while the diagonal terms equal $f(1)$. Hence

$$S_f(X) \leq N f(1).$$

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Combining the inequalities gives the standard linear-programming bound

$$N \leq \frac{f(1)}{f_0}.$$

The proof is important because it separates a universal positivity inequality from a polynomial design problem. Once a suitable f is found, the bound follows from a short certificate. This method proves the exact kissing numbers in dimensions 8 and 24 through the exceptional algebraic structure of the E_8 and Leech lattices (Odlyzko & Sloane, 1979, pp. 210-214).

MUSIN'S EXTENSION IN THREE DIMENSIONS

The classical sign condition is too restrictive to yield the sharp three-dimensional bound directly. Musin instead uses a degree-nine polynomial whose Legendre coefficients are nonnegative but whose values may be positive for inner products close to -1 . In Legendre form,

$$f = P_0 + \frac{8}{5}P_1 + \frac{87}{25}P_2 + \frac{33}{20}P_3 + \frac{49}{25}P_4 + \frac{1}{10}P_5 + \frac{8}{25}P_9.$$

Because the coefficient of P_0 is 1, Schoenberg positivity gives

$$S_f(X) \geq N^2.$$

Musin proves that $f(t) < 0$ on most of the interval $[-1, 1/2]$. Positive off-diagonal contributions can occur only when t is sufficiently close to -1 , meaning that the corresponding point lies in a small spherical cap around the antipode of a fixed point. For each $x_i \in X$, define

$$S_i(X) = \sum_{j=1}^N f(\langle x_i, x_j \rangle).$$

A local spherical argument shows that at most four code points can lie in the exceptional antipodal cap and then bounds every possible local contribution by

$$S_i(X) < 13.$$

Summing over i yields

$$S_f(X) = \sum_{i=1}^N S_i(X) < 13N.$$

Therefore,

$$N^2 \leq S_f(X) < 13N,$$

so $N < 13$ and hence $N \leq 12$. Combined with the FCC construction, this proves $k(3) = 12$.

The argument is correctly described as an extended linear-programming proof rather than a purely linear-programming proof. The polynomial supplies a global algebraic inequality, but the sharp constant depends on a separate geometric bound for the exceptional positive terms (Musin, 2006, pp. 377-384).

COMPARISON OF METHODS

The methods differ first in what they certify. The FCC lattice is a direct proof: it exhibits twelve neighbors and proves a lower bound. The geometric proof and Musin's argument is an indirect proof: they prove that a thirteenth neighbor cannot exist.

The geometric method works directly with the metric structure of S^2 . Its basic objects-spherical caps, contact graphs, triangles, and polygons-have immediate visual interpretations. It can exploit features special to two-dimensional surfaces, particularly planarity and Euler's formula. Yet its proof burden is concentrated in case reduction. Each admissible contact graph must satisfy nonlinear trigonometric inequalities, and the number of candidate structures grows quickly. The method is rigorous but difficult to generalize to higher dimensions.

The linear-programming method replaces most configuration-specific geometry with a universal positive-semidefinite kernel. The inequality

$$\sum_{i,j} P_k(\langle x_i, x_j \rangle) \geq 0$$

holds for every finite code, so a single auxiliary polynomial can bound all configurations at once. This changes the proof problem from classifying point sets to finding coefficients that satisfy positivity and sign constraints. The resulting certificate is concise, checkable, and compatible with computation. It also extends to Gegenbauer polynomials in arbitrary dimension and, through semidefinite programming, to stronger constraints involving triples of points (Bachoc & Vallentin, 2008, pp. 909-924).

Nevertheless, the three-dimensional proof demonstrates a limitation of the basic Delsarte framework. A polynomial that is globally nonpositive on every admissible off-diagonal inner product does not produce the sharp value twelve. Musin obtains sharpness by allowing a controlled positive interval and then solving the resulting local geometric problem.

From the standpoint of future research, the optimization framework is more generalizable. Semidefinite programming strengthens two-point bounds by incorporating matrix-valued positive kernels and three-point

distributions, and it has produced improved upper bounds in several unsolved dimensions (Bachoc & Vallentin, 2008, pp. 909-924). Geometric insight remains indispensable, however, both for interpreting equality cases and for converting numerical optimization output into rigorous analytic certificates.

OPEN PROBLEMS AND FUTURE DIRECTIONS

Exact kissing numbers are known only in dimensions 1, 2, 3, 4, 8, and 24. In other dimensions, lower bounds come from explicit codes or lattices, while upper bounds come from linear, semidefinite, and related optimization methods. Recent improvements in dimensions seventeen through twenty-one show that even long-standing lower bounds can change when new nonlattice constructions are found (Cohn & Li, 2024).

Three directions are especially important. First, sharper semidefinite hierarchies may capture higher-order correlations that two-point methods omit. Second, computer-assisted certificates must be converted into exact, independently verifiable mathematics through rational reconstruction, interval arithmetic, or formal proof. Third, classification questions remain even when the numerical kissing number is known: one may ask whether an optimal code is unique up to orthogonal transformations or whether several geometrically distinct extremizers exist.

CONCLUSION

The equality $k(3) = 12$ contains two logically separate claims: twelve spheres can touch a central sphere, and thirteen cannot. The FCC lattice proves the first claim through its twelve minimal vectors. The second requires a genuine upper-bound argument. Classical geometry achieves it through spherical contact structures, while Musin's proof combines Schoenberg positivity, an auxiliary polynomial, and local spherical analysis. The comparison shows that geometric methods provide structural understanding and that optimization methods provide transferable global certificates. Their interaction, rather than either method alone, best explains both the solution in three dimensions and the continuing difficulty of the kissing number problem in higher dimensions.

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